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Positive solutions for a class of fractional differential equations at resonance

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Abstract

In this paper, by using the Leggett-Williams norm-type theorem, we consider a *m*-point boundary value problem for a class of fractional differential equations at resonance. A new result on the existence of solutions for above problem is obtained.

MSC: 34A08; 34B15

Keywords: fractional differential equation; *m*-point boundary value problem; Leggett-Williams norm-type theorem; resonance

1 Introduction

The subject of fractional calculus has gained significant interest and been a valuable tool for both science and engineering (see [1–3]). In recent years, the fractional boundary value problems (FBVPs for short) have been considered by many authors (see [4–10] and the references therein). For example, Bai studied a FBVP at non-resonance with $1 < \alpha \le 2$ (see [10]). FBVPs at resonance were studied by Kosmatov (see [11]) and Jiang (see [12]). But the positive solutions for FBVPs at resonance were studied very few. In [13], Yang and Wang considered the positive solutions of the following FBVP:

$$\begin{cases} -D_{0^+}^{\alpha} x(t) = f(t, x(t)), & t \in [0,1], \\ x(0) = 0, & x'(0) = x'(1). \end{cases}$$

In [14], Chen and Tang studied the positive solution of FBVP as follows:

$$\begin{cases} D_{0^+}^{\alpha} x(t) = f(t, x(t)), & t \in [0,+\infty), \\ x(0) = x'(0) = x''(0) = 0, & D_{0^+}^{\alpha-1} x(0) = \lim_{t\to+\infty} D_{0^+}^{\alpha-1} x(t). \end{cases}$$

However, to the best of our knowledge, the fractional differential equations with m-point boundary conditions at resonance have not been considered. Motivated by the papers above, we consider the existence of positive solutions for a m-point FBVP of the form

$$\begin{cases} -D_{0^+}^{\alpha} x(t) = f(t, x(t)), & t \in [0,1], \\ x'(0) = 0, & x(1) = \sum_{i=1}^{m-2} \beta_i x(\eta_i), \end{cases} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ denotes the standard Caputo fractional differential operator of order α, $1 < \alpha \le 2$, $\beta_i \in \mathbb{R}^+$, $\sum_{i=1}^{m-2} \beta_i = 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous. Obviously, FBVP (1.1) happens to be at resonance under the condition $\sum_{i=1}^{m-2} \beta_i = 1$.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on the existence of positive solutions for FBVP (1.1) under some restrictions of f, basing on the coincidence degree theory due to [15]. Finally, in Section 4, an example is given to illustrate the main result.

2 Preliminaries

For convenience of the reader, we present some definitions, notations, and preliminary statements, which can be found in [2, 16, 17].

Let X and Y be real Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator with index zero, where the index of a Fredholm operator L is defined by

$$\operatorname{Index} L := \dim \operatorname{Ker} L - \dim \operatorname{Coker} L.$$

Suppose $P : X \to X$, $Q : Y \to Y$ be continuous linear projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Ker} Q = \operatorname{Im} L,$$
$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus, we see that

$$L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$$

is invertible. We denote the inverse by K_P. Moreover, by virtue of $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Then we know that the operator equation $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx,$$

where $N : X \to Y$ be a nonlinear operator.

If Ω is an open bounded subset of X such that $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, then the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN : \overline{\Omega} \to Y$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Let C be a cone in X. Then C induces a partial order in X by

$$x \le y \quad \text{if and only if} \quad y - x \in C.$$

Lemma 2.1 (see [15]) *Let C be a cone in X. Then for every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \ge \sigma(u)\|x\|$$

for all $x \in C$.

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Set

$$\Psi := P + JQN + K_P(I - Q)N$$

and

$$\Psi_\gamma := \Psi \circ \gamma.$$

Lemma 2.2 (see [15]) *Let C be a cone in X and Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the following conditions are satisfied*:

1. $L : \operatorname{dom} L \subset X \to Y$ *be a Fredholm operator of index zero and* $N : X \to Y$ *be L-compact on every bounded subset of X*,
2. $Lx \neq \lambda Nx$ *for every* $(x,\lambda) \in [C \cap \partial\Omega_2 \cap \operatorname{dom} L] \times (0,1)$,
3. γ *maps subsets of* $\overline{\Omega}_2$ *into bounded subsets of* C,
4. $\deg([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$,
5. *there exists* $u_0 \in C \setminus \{0\}$ *such that* $\|x\| \leq \sigma(u_0)\|\Psi x\|$ *for* $x \in C(u_0) \cap \partial\Omega_1$, *where* $C(u_0) = \{x \in C : \mu u_0 \leq x \text{ for some } \mu > 0\}$ *and* $\sigma(u_0)$ *is such that* $\|x + u_0\| \geq \sigma(u_0)\|x\|$ *for every* $x \in C$,
6. $(P + JQN)\gamma(\partial\Omega_2) \subset C$,
7. $\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.

Then the equation $Lx = Nx$ has at least one solution in $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2.3 (see [17]) The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function x is given by

$$I_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided that the right side integral is pointwise defined on $(0, +\infty)$.

Definition 2.4 (see [17]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function x is given by

$$D_{0^+}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s)\, ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.5 (see [18]) *For $\alpha > 0$, the general solution of the Caputo fractional differential equation*

$$D_{0^+}^{\alpha} x(t) = 0$$

is

$$x(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

Lemma 2.6 (see [18]) *Suppose that $D_{0^+}^{\alpha} x \in C[0,1]$, $\alpha > 0$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

In this paper, we take $X = Y = C[0,1]$ with the norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$.

Define the operator $L : \operatorname{dom} L \subset X \to Y$ by

$$Lx = -D_{0^+}^\alpha x, \tag{2.1}$$

where

$$\operatorname{dom} L = \left\{ x \in X : D_{0^+}^\alpha x \in Y, x'(0) = 0, x(1) = \sum_{i=1}^{m-2} \beta_i x(\eta_i) \right\}.$$

Let $N : X \to Y$ be the Nemytskii operator

$$Nx(t) = f\bigl(t, x(t)\bigr), \quad \forall t \in [0,1].$$

Then FBVP (1.1) is equivalent to the operator equation

$$Lx = Nx, \quad x \in \operatorname{dom} L.$$

3 Main result

In this section, a theorem on the existence of positive solutions for FBVP (1.1) will be given.

For simplicity of notation, we set

$$l_i(s) = \begin{cases} (1-s)^{\alpha-1} - (\eta_i - s)^{\alpha-1}, & 0 \le s \le \eta_i \le 1, \\ (1-s)^{\alpha-1}, & 0 \le \eta_i \le s \le 1, \end{cases}$$

and

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha \\ \quad + \frac{\alpha(\Gamma(\alpha+2)-1+(\alpha+1)t^\alpha)}{\Gamma(\alpha+2)\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i l_i(s), & 0 \le t \le s \le 1, \\ \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1} \\ \quad + \frac{\alpha(\Gamma(\alpha+2)-1+(\alpha+1)t^\alpha)}{\Gamma(\alpha+2)\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i l_i(s), & 0 \le s \le t \le 1. \end{cases}$$

Obviously, $\max_{0\le s\le 1} \sum_{i=1}^{m-2} \beta_i l_i(s) \le 1$. We denote

$$\kappa := \min\left\{ 1, \frac{\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)}{\alpha}, \frac{1}{\max_{t,s\in[0,1]} G(t,s)} \right\}.$$

Thus, one has

$$1 - \frac{\kappa\alpha\sum_{i=1}^{m-2}\beta_i l_i(s)}{\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)} \ge 0, \quad 1 - \kappa G(t,s) \ge 0. \tag{3.1}$$

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that:*

(H_1) *there exist nonnegative functions $a, b \in X$ with $\frac{\Gamma(\alpha+1)}{2} b_1 < 1$ such that*

$$\bigl|f(t,u)\bigr| \le a(t) + b(t)|u|, \quad \forall t \in [0,1], u \in \mathbb{R},$$

where $b_1 = \|b\|_\infty$,

(H_2) *there exists a constant* $B > 0$ *such that*

$$uf(t,u) < 0, \quad \forall t \in [0,1], |u| > B,$$

(H_3) $f(t,u) > -\kappa u$, *for all* $(t,u) \in [0,1] \times [0,\infty)$,

(H_4) *there exist* $r \in (0,+\infty)$, $t_0 \in [0,1]$, $M \in (0,1)$ *and continuous function* $h : (0,r] \to [0,\infty)$ *such that* $f(t,u) \geq h(u)$ *for all* $t \in [0,1]$, $u \in (0,r]$, *and* $\frac{h(u)}{u}$ *is non-increasing on* $(0,r]$ *with*

$$\frac{h(r)}{r}\int_0^1 G(t_0,s)\,ds \geq \frac{1-M}{M}.$$

Then FBVP (1.1) *has at least one solution in* X.

Now, we begin with some lemmas that are useful in what follows.

Lemma 3.2 *Let* L *be defined by* (2.1), *then*

$$\operatorname{Ker} L = \{x \in X : x(t) = c, \forall t \in [0,1], c \in \mathbb{R}\}, \tag{3.2}$$

$$\operatorname{Im} L = \left\{y \in Y : \sum_{i=1}^{m-2}\beta_i\int_0^1 l_i(s)y(s)\,ds = 0\right\}. \tag{3.3}$$

Proof By Lemma 2.5, $D_{0^+}^{\alpha}x(t) = 0$ has solution

$$x(t) = c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

Combining with the boundary conditions of FBVP (1.1), one sees that (3.2) holds.

For $y \in \operatorname{Im} L$, there exists $x \in \operatorname{dom} L$ such that $y = Lx \in Y$. By Lemma 2.6, we have

$$x(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)\,ds + c_0 + c_1 t, \quad c_0, c_1 \in \mathbb{R}.$$

Then we get

$$x'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}y(s)\,ds + c_1.$$

By the boundary conditions of FBVP (1.1), we see that y satisfies

$$\int_0^1 (1-s)^{\alpha-1}y(s)\,ds = \sum_{i=1}^{m-2}\beta_i\int_0^{\eta_i}(\eta_i - s)^{\alpha-1}y(s)\,ds.$$

That is,

$$\sum_{i=1}^{m-2}\beta_i\int_0^1 l_i(s)y(s)\,ds = 0. \tag{3.4}$$

On the other hand, suppose $y \in Y$ and satisfies (3.4). Let $x(t) = -I_{0^+}^{\alpha}y(t) + x(0)$, then $x \in \operatorname{dom} L$ and $D_{0^+}^{\alpha}x(t) = -y(t)$. Thus, $y \in \operatorname{Im} L$. Hence (3.3) holds. The proof is complete. $\square$

Lemma 3.3 *Let L be defined by* (2.1), *then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : X \to X$ and $Q : Y \to Y$ can be defined as*

$$Px(t) = \int_0^1 x(s)\,ds, \quad \forall t \in [0,1],$$

$$Qy(t) = \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s)y(s)\,ds, \quad \forall t \in [0,1].$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P y(t) = \int_0^1 k(t,s)y(s)\,ds, \quad \forall t \in [0,1],$$

where

$$k(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha, & 0 \le t \le s \le 1, \\ \frac{1}{\Gamma(\alpha+1)}(1-s)^\alpha - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, & 0 \le s \le t \le 1. \end{cases} \tag{3.5}$$

Proof Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2 x = Px$. It follows from $x = (x - Px) + Px$ that $X = \operatorname{Ker} P + \operatorname{Ker} L$. By a simple calculation, one obtain $\operatorname{Ker} L \cap \operatorname{Ker} P = \{0\}$. Thus, we get

$$X = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

For $y \in Y$, we have

$$Q^2 y = Q(Qy) = Qy \cdot \frac{\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s)\,ds = Qy.$$

Let $y = (y - Qy) + Qy$, where $y - Qy \in \operatorname{Ker} Q$, $Qy \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2 y = Qy$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{0\}$. Then one has

$$Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus, we obtain

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \dim \operatorname{Coker} L = 1.$$

That is, L is a Fredholm operator of index zero.

Now, we will prove that K_P is the inverse of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P}$. In fact, for $y \in \operatorname{Im} L$, we have

$$K_P y(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,ds + c_0, \tag{3.6}$$

where

$$c_0 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 (1-s)^\alpha y(s)\,ds.$$

It is easy to see that $LK_Py = y$. Moreover, for $x \in \operatorname{dom} L \cap \operatorname{Ker} P$, we get $x'(0) = 0$ and

$$\begin{aligned} K_PLx(t) &= I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) - I_{0^+}^{\alpha+1} D_{0^+}^{\alpha} x(t)|_{t=1} \\ &= x(t) - x(0) - \int_0^1 \bigl(x(s) - x(0)\bigr)\, ds \\ &= x - Px. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we know that $K_P = (L|_{\operatorname{dom} L\cap \operatorname{Ker} P})^{-1}$. The proof is complete. □

Lemma 3.4 *Assume $\Omega \subset X$ is an open bounded subset such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$, *then N is L-compact on $\overline{\Omega}$.*

Proof By the continuity of f, we see that $QN(\overline{\Omega})$ and $K_P(I - Q)N(\overline{\Omega})$ are bounded. That is, there exist constants $A, B > 0$ such that $|(I - Q)Nx| \leq A$ and $|K_P(I - Q)Nx| \leq B$, $\forall x \in \overline{\Omega}$, $t \in [0, 1]$. Thus, one need only prove that $K_P(I - Q)N(\overline{\Omega}) \subset X$ is equicontinuous.

Let $K_{P,Q} = K_P(I - Q)N$, for $0 \leq t_1 < t_2 \leq 1$, $x \in \overline{\Omega}$, we get

$$\begin{aligned} &\bigl|(K_{P,Q}x)(t_2) - (K_{P,Q}x)(t_1)\bigr| \\ &\quad\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_2} (t_2 - s)^{\alpha-1}(I - Q)Nx(s)\, ds - \int_0^{t_1} (t_1 - s)^{\alpha-1}(I - Q)Nx(s)\, ds \right| \\ &\quad\leq \frac{A}{\Gamma(\alpha)} \left[\int_0^{t_1} (t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\, ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1}\, ds \right] \\ &\quad= \frac{A}{\Gamma(\alpha + 1)} \bigl(t_2^{\alpha} - t_1^{\alpha}\bigr). \end{aligned}$$

Since t^{α} is uniformly continuous on $[0, 1]$, we see that $K_{P,Q}N(\overline{\Omega}) \subset X$ is equicontinuous. Thus, we see that $K_{P,Q}N : \overline{\Omega} \to X$ is compact. The proof is completed. □

Lemma 3.5 *Suppose* (H_1) *and* (H_2) *hold, then the set*

$$\Omega_0 = \bigl\{x \in \operatorname{dom} L : Lx = \lambda Nx, \lambda \in (0, 1)\bigr\}$$

is bounded.

Proof Take $x \in \Omega_0$, then $Nx \in \operatorname{Im} L$. By (3.2), we have

$$\sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f\bigl(s, x(s)\bigr)\, ds = 0.$$

Then, by the integral mean value theorem, there exists a constant $\xi \in (0, 1)$ such that $f(\xi, x(\xi)) = 0$. So, from (H_2), we get $|x(\xi)| \leq B$. By Lemma 2.6, one has

$$\begin{aligned} x(t) &= x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} D_{0^+}^{\alpha} x(s)\, ds, \\ x(\xi) &= x(0) + \frac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} D_{0^+}^{\alpha} x(s)\, ds. \end{aligned}$$

Thus, we get

$$x(t) - x(\xi) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0^+}^{\alpha} x(s)\,ds - \frac{1}{\Gamma(\alpha)} \int_0^{\xi} (\xi-s)^{\alpha-1} D_{0^+}^{\alpha} x(s)\,ds,$$

which together with (H_1) implies that

$$\begin{aligned} |x(t)| &\le |x(\xi)| + \frac{1}{\Gamma(\alpha)} \|D_{0^+}^{\alpha} x\|_\infty \cdot \frac{1}{\alpha}\left(t^\beta + \xi^\beta\right) \\ &\le B + \frac{2}{\Gamma(\alpha+1)} \|D_{0^+}^{\alpha} x\|_\infty \\ &\le B + \frac{2}{\Gamma(\alpha+1)} \cdot \max_{t\in[0,1]} \left|f\left(t, x(t)\right)\right| \\ &\le B + \frac{2}{\Gamma(\alpha+1)} \left(\|a\|_\infty + b_1\|x\|_\infty\right), \quad \forall t \in [0,1]. \end{aligned}$$

That is,

$$\|x\|_\infty \le B + \frac{2}{\Gamma(\alpha+1)} \left(\|a\|_\infty + b_1\|x\|_\infty\right).$$

In view of $\frac{2}{\Gamma(\alpha+1)} b_1 < 1$, there exists a constant $D_2 > 0$ such that

$$\|x\|_\infty \le D_2.$$

Hence, Ω_0 is bounded. The proof is complete. □

Proof of Theorem 3.1 Set $C = \{x \in X : x(t) \ge 0, t \in [0,1]\}$, $\Omega_1 = \{x \in X : r > |x(t)| > M\|x\|_\infty, t \in [0,1]\}$, and $\Omega_2 = \{x \in X : \|x\|_\infty < R\}$, where $R = \max\{B, D_2\}$. Clearly, Ω_1, Ω_2 are open bounded subsets of X and

$$\overline{\Omega}_1 = \left\{x \in X : r \ge |x(t)| \ge M\|x\|_\infty, t \in [0,1]\right\} \subset \Omega_2.$$

From Lemma 3.3, Lemma 3.4, and Lemma 3.5, we see that the conditions (1) and (2) of Lemma 2.2 are satisfied.

Let $\gamma x(t) = |x(t)|$ for $x \in X$ and $J = I$. One can see that γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C, which means that the condition (3) of Lemma 2.2 holds.

For $x \in \operatorname{Ker} L \cap \Omega_2$, we have $x(t) \equiv c$. Let

$$H(c,\lambda) = c - \lambda|c| - \frac{\lambda\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f\left(s, |c|\right) ds.$$

From $H(c,\lambda) = 0$, one has $c \ge 0$. Moreover, if $H(R,\lambda) = 0$, we get

$$0 \le R(1-\lambda) = \frac{\lambda\alpha}{\sum_{i=1}^{m-2} \beta_i(1-\eta_i^\alpha)} \sum_{i=1}^{m-2} \beta_i \int_0^1 l_i(s) f(s,R)\,ds,$$

which contradicts (H_2). Thus $H(c,\lambda) \neq 0$ for $x \in \partial\Omega_2$, $\lambda \in [0,1]$. Hence

$$\begin{aligned}
&\deg\bigl(\bigl[I-(P+JQN)\gamma\bigr]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0\bigr) \\
&\quad = \deg\bigl(H(c,1), \operatorname{Ker} L \cap \Omega_2, 0\bigr) \\
&\quad = \deg\bigl(H(c,0), \operatorname{Ker} L \cap \Omega_2, 0\bigr) \\
&\quad = \deg(I, \operatorname{Ker} L \cap \Omega_2, 0) \\
&\quad = 1.
\end{aligned}$$

So, the condition (4) of Lemma 2.2 holds.

Let $x \in \overline{\Omega}_2 \setminus \Omega_1$, $t \in [0,1]$, we have

$$\begin{aligned}
\Psi_\gamma x(t) &= \int_0^1 \bigl|x(s)\bigr|\,ds + \frac{\alpha}{\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)}\sum_{i=1}^{m-2}\beta_i\int_0^1 l_i(s)f\bigl(s,\bigl|x(s)\bigr|\bigr)\,ds \\
&\quad + \int_0^1 k(t,s)\Biggl[f\bigl(s,\bigl|x(s)\bigr|\bigr) - \frac{\alpha}{\sum_{i=1}^{m-2}\beta_i(1-\eta_i^\alpha)}\sum_{i=1}^{m-2}\beta_i\int_0^1 l_i(\tau)f\bigl(\tau,\bigl|x(\tau)\bigr|\bigr)\,d\tau\Biggr]\,ds \\
&= \int_0^1 \bigl|x(s)\bigr|\,ds + \int_0^1 G(t,s)f\bigl(s,\bigl|x(s)\bigr|\bigr)\,ds,
\end{aligned}$$

which together with (H_3) and (3.1) yields

$$\Psi_\gamma x(t) \geq \int_0^1 \bigl|x(s)\bigr|\,ds - \kappa\int_0^1 G(t,s)\bigl|x(s)\bigr|\,ds = \int_0^1 \bigl(1-\kappa G(t,s)\bigr)\bigl|x(s)\bigr|\,ds \geq 0.$$

Thus, the condition (7) of Lemma 2.2 holds. In addition, we can prove the condition (6) of Lemma 2.2 holds too by a similar process.

Finally, we will show that the condition (5) of Lemma 2.2 is satisfied. Let $u_0(t) \equiv 1$, $t \in [0,1]$, then $u_0 \in C \setminus \{0\}$, $C(u_0) = \{x \in C : x(t) > 0, t \in [0,1]\}$ and we can take $\sigma(u_0) = 1$. For $x \in C(u_0) \cap \partial\Omega_1$, we have $x(t) > 0$, $t \in [0,1]$, $0 < \|x\|_\infty \leq r$, and $x(t) \geq M\|x\|_\infty$, $t \in [0,1]$. So, from (H_4), we obtain

$$\begin{aligned}
\Psi x(t_0) &= \int_0^1 x(s)\,ds + \int_0^1 G(t_0,s)f\bigl(s,x(s)\bigr)\,ds \\
&\geq M\|x\| + \int_0^1 G(t_0,s)h\bigl(x(s)\bigr)\,ds \\
&= M\|x\| + \int_0^1 G(t_0,s)\frac{h(x(s))}{x(s)}x(s)\,ds \\
&\geq M\|x\| + \frac{h(r)}{r}\int_0^1 G(t_0,s)x(s)\,ds \\
&\geq M\|x\| + \frac{h(r)}{r}\int_0^1 G(t_0,s)M\|x\|\,ds \\
&\geq M\|x\| + (1-M)\|x\| \\
&= \|x\|.
\end{aligned}$$

Then the condition (5) of Lemma 2.2 holds.

Consequently, by Lemma 2.2, the equation $Lx = Nx$ has at least one solution $x^* \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$. Namely, FBVP (1.1) has at least one positive solution in X. The proof is complete. □

4 Example

We consider the following FBVP:

$$\begin{cases} -D_{0^+}^{\frac{3}{2}} x(t) = \sin t - \frac{1}{10}x(t) + 10 + \sin x(t), & t \in [0,1], \\ x'(0) = 0, \qquad x(1) = x(\frac{1}{2}). \end{cases} \tag{4.1}$$

Thus, we have

$$l(s) = \begin{cases} \sqrt{1-s} - \sqrt{\frac{1}{2} - s}, & 0 \le s \le \frac{1}{2}, \\ \sqrt{1-s}, & \frac{1}{2} \le s \le 1, \end{cases}$$

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} + \frac{3(\Gamma(\frac{7}{2})-1+(\frac{5}{2})t^{\frac{3}{2}})}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})} l(s), & 0 \le t \le s \le 1, \\ \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} - \frac{1}{\Gamma(\frac{3}{2})}(t-s)^{\frac{1}{2}} + \frac{3(\Gamma(\frac{7}{2})-1+(\frac{5}{2})t^{\frac{3}{2}})}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})} l(s), & 0 \le s \le t \le 1. \end{cases}$$

Moreover, $f(t,u) \ge 8 - \frac{1}{10}u \ge -\frac{1}{4}u$ for all $u \ge 0$, and $l(s) \le 1$, $G(t,s) \le 4$, $\kappa = -\frac{1}{4}$. So, we can find that (H_1), (H_2), (H_3) hold. Next, we take $t_0 = 0$, $h(x) = x$, and $M = \frac{2}{3}$, thus $G(0,s) = \frac{1}{\Gamma(\frac{5}{2})}(1-s)^{\frac{3}{2}} + \frac{3(\Gamma(\frac{7}{2})-1)}{2\Gamma(\frac{7}{2})(1-(\frac{1}{2})^{\frac{3}{2}})} l(s)$, $0 \le s \le 1$, and $\int_0^1 G(0,s)\,ds = 1$. Then (H_4) is satisfied. According to the above points, by Theorem 3.1, we can conclude that FBVP (4.1) has at least one positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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